

# Avoider-Enforcer star games

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## Abstract

In this paper, we study  $(1 : b)$  Avoider-Enforcer games played on the edge set of the complete graph  $K_n$ , on  $n$  vertices. In particular, we give explicit winning strategies for both players in the  $k$ -star game, for constant  $k \geq 2$  under both strict and monotone rules. We also study two more related monotone games.

## 1 Introduction

Let  $V$  be a finite set and let  $\mathcal{F} \subseteq 2^V$ . Consider a hypergraph  $\mathcal{H} = (V, \mathcal{F})$  with vertex set  $V$  and edge set  $\mathcal{F}$ . The set  $V$  is called the *board* and  $\mathcal{F}$  the family of *losing sets*. Two players, Avoider and Enforcer, take turns in claiming unoccupied vertices of  $V$  until all vertices are claimed and Avoider starts the game. Avoider's goal in the game is to avoid claiming all the elements of any losing set in  $\mathcal{F}$ , while Enforcer's goal is to force him to do so before the end of the game. Avoider-Enforcer games can be played by two different sets of rules [10].

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Let  $a$  and  $b$  be positive integers called the *biases* of Avoider, respectively Enforcer. By the first set of rules, in the  $(a : b)$  Avoider-Enforcer game, Avoider claims exactly  $a$  vertices and Enforcer claims exactly  $b$  vertices per move. This set of rules is called the *strict rules*, and games played by these rules are referred to as *strict games*. By the second set of rules, the *monotone rules*, Avoider and Enforcer claim at least  $a$ , respectively at least  $b$  vertices, per move, and games played by monotone rules are called *monotone games*. In any move of either player, if there are less unclaimed vertices than what a player should claim in his turn, he must claim all the remaining vertices and the game is then over. The game is an *Enforcer's win* if, at any point of the game Avoider has claimed all the vertices of at least one losing set  $F \in \mathcal{F}$ . Otherwise, the game is an *Avoider's win*.

One of the main advantages of the monotone rules is that they are bias monotone [10]. In monotone Avoider-Enforcer games, if the  $(a : b)$  game is an Enforcer's win, then the  $(a + 1 : b)$  and  $(a : b - 1)$  games are also won by Enforcer. Similarly, if the  $(a : b)$  game is an Avoider's win, then the  $(a : b + 1)$  and  $(a - 1 : b)$  games are also won by Avoider.

When  $a = b = 1$ , we call such games *unbiased*. Unbiased Avoider-Enforcer games were studied e.g. in [2], [4], [13], [14], [15], [16]. In an unbiased game, it is also interesting to see how fast Enforcer can force Avoider to lose, or to see how long can Avoider defend himself before losing. These type of problems were considered in [1], [8].

In [13], Lu proved that Beck's [3] generalization of the Erdős-Selfridge criterion [7] gave sufficient conditions for an Avoider's win in the  $(1 : 1)$  Avoider-Enforcer game. In [11], Hefetz, Krivelevich and Szabó gave a general winning criterion for Avoider in  $(a : b)$  Avoider-Enforcer games played by both sets of rules. This criterion takes only Avoider's bias into account. In [5] a new criterion for Avoider's win in both strict and monotone  $(a : b)$  games on  $\mathcal{H}$  is introduced, which depends on both biases  $a$  and  $b$ . Note, however, these criteria are non-constructive and do not give the strategy of the winning player.

The focus of our research is on  $(1 : b)$  Avoider-Enforcer games, played by both monotone and strict rules on a graph where the board is the set of edges of the complete graph on  $n$  vertices, i.e.  $V = E(K_n)$ . This type of game appears frequently in the literature (see, for example [9], [10] and [11]).

We follow the terminology for strict Avoider-Enforcer games introduced by Hefetz et al. in [11]. The *upper threshold bias*  $f_{\mathcal{H}}^+$  is the smallest integer such that for every integer  $b$ ,  $b > f_{\mathcal{H}}^+$ , the  $(1 : b)$  game on  $\mathcal{H}$  is an Avoider's win. The *lower threshold bias*  $f_{\mathcal{H}}^-$  is the largest integer such that for every integer  $b$ ,  $b \leq f_{\mathcal{H}}^-$ , the  $(1 : b)$  game on  $\mathcal{H}$  is an Enforcer's win. The inequality  $f_{\mathcal{H}}^- \leq f_{\mathcal{H}}^+$  always holds, but the upper and lower threshold biases can be close to each other in the case of some games, like for example in Connectivity game [9] or be far apart. When  $f_{\mathcal{H}}^- = f_{\mathcal{H}}^+$  we call this number  $f_{\mathcal{H}}$  and refer to it as the *threshold bias* of the game  $\mathcal{H}$ . This threshold bias may not exist for some games [10].

For monotone  $(1 : b)$  Avoider-Enforcer game on  $\mathcal{H}$ , there is a unique *monotone threshold bias*  $f_{\mathcal{H}}^{mon}$  defined as the largest non-negative integer value such that for every integer  $b$  s.t.  $b \leq f_{\mathcal{H}}^{mon}$ , the game is an Enforcer's win [10]. As it turns out,  $f_{\mathcal{H}}^- \leq f_{\mathcal{H}}^{mon} \leq f_{\mathcal{H}}^+$  does not hold for all  $(1 : b)$  Avoider-Enforcer games. The results in the Connectivity Game,

Hamiltonicity Game, Perfect matching game, Minimum degree  $k$  game (for  $k \geq 1$ ),  $k$ -connectivity game (for  $k \geq 1$ ) and  $k$ -edge connectivity game (for  $k \geq 1$ ) show that this is not true in general (see [10] and [11]).

We are interested in  $(1 : b)$  Avoider-Enforcer games where Avoider wants to avoid claiming a copy of some small fixed graph  $G$ . This problem was studied in [10] for some graphs  $G$ . Let  $\mathcal{K}_G$  denote the hypergraph whose losing sets are edges of all the copies of  $G$  in  $K_n$ . In [10] the authors analysed games where  $G$  is  $K_3$  and  $P_3$  respectively and gave the thresholds for both the monotone and strict game  $\mathcal{K}_{P_3}$  and for the monotone game  $\mathcal{K}_{K_3}$ . They showed that

$$f_{\mathcal{K}_{P_3}}^{mon} = \binom{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor - 1, f_{\mathcal{K}_{P_3}}^+ = \binom{n}{2} - 2, f_{\mathcal{K}_{P_3}}^- = \Theta(n^{\frac{3}{2}}) \text{ and } f_{\mathcal{K}_{K_3}}^{mon} = \Theta(n^{\frac{3}{2}}).$$

Bednarska-Bzdęga in [5] showed that  $f_{\mathcal{K}_{K_3}}^- = \Omega(n^{\frac{1}{2}})$ . Clemens et al. in [6] showed that in a monotone game where losing sets are the edges of all  $P_4$ , the threshold bias is  $f_{\mathcal{K}_{P_4}}^{mon} = \frac{1}{2} \binom{n}{2} - \frac{n}{2} \left( \frac{1}{\sqrt{2}} - o(1) \right)$ . Moreover, in [10], the authors have conjectured that in general  $f_{\mathcal{K}_G}^+$  and  $f_{\mathcal{K}_G}^-$  are not of the same order and asked about the strategies in games where  $G$  is some fixed graph on more than 4 vertices. Bednarska-Bzdęga established in [5] general upper and lower bounds on  $f_{\mathcal{K}_G}^+$ ,  $f_{\mathcal{K}_G}^-$  and  $f_{\mathcal{K}_G}^{mon}$  for every fixed graph  $G$ , but these bounds are not tight.

In the present paper, we study the game  $\mathcal{K}_G$  where  $G$  is a  $k$ -star  $K_{1,k}$ , for some fixed  $k \geq 2$  and denote it by  $\mathcal{S}_k$ . We call the game  $k$ -star game.

In order to state our main result, we have to introduce some functions: let us define  $r = r(n, b)$  by  $1 \leq r \leq b + 1$  and  $\binom{n}{2} \equiv r \pmod{b + 1}$ . Note that  $r$  is the number of edges that Avoider is allowed to choose from in his last move when playing the strict  $(1 : b)$  game.

Let

$$e_{n,k}^+ = \max \left\{ b : r(b + 1) < \frac{1}{8} \frac{n^k}{(2b)^{k-2}} \right\}, \text{ and} \\ e_{n,k}^- = \max \left\{ b' : \forall b, \frac{1}{4} n^{\frac{k+1}{k}} \leq b \leq b' : r(b + 1) < \frac{1}{8} \frac{n^k}{(2b)^{k-2}} \right\}.$$

The main result in our paper is the following theorem.

**Theorem 1.1.** *In  $(1 : b)$   $k$ -star game  $\mathcal{S}_k$ ,  $k \geq 2$ , we have*

- (i)  $f_{\mathcal{S}_k}^{mon} = \Theta(n^{\frac{k}{k-1}})$ ,
- (ii)  $e_{n,k}^+ \leq f_{\mathcal{S}_k}^+ = O(n^{\frac{k}{k-1}})$  holds for all values of  $n$ , and  $f_{\mathcal{S}_k}^+ = \Theta(n^{\frac{k}{k-1}})$  holds for infinitely many values of  $n$ ,
- (iii)  $\max\{\frac{1}{2} n^{\frac{k+1}{k}}, e_{n,k}^-\} \leq f_{\mathcal{S}_k}^- = O(n^{\frac{k+1}{k}} \log n)$  holds for all values of  $n$ , and  $f_{\mathcal{S}_k}^- = \Theta(n^{\frac{k+1}{k}})$  holds for infinitely many values of  $n$ .

We also consider two more monotone games similar to the  $k$ -star game. Let the *double star*  $\mathcal{S}_{k,k}$  be a graph on  $2k$  vertices  $u, u_1, \dots, u_{k-1}, v, v_1, \dots, v_{k-1}$  such that the edge set of  $\mathcal{S}_{k,k}$  is  $\{(uv)\} \cup \{(uu_i) : 1 \leq i \leq k-1\} \cup \{(vv_i) : 1 \leq i \leq k-1\}$  (see Figure 1) and let  $\mathcal{K}_{\mathcal{S}_{k,k}}$  be the hypergraph of the game.

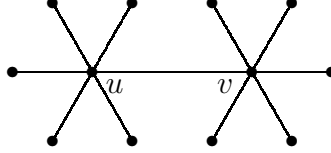


Figure 1:  $\mathcal{S}_{6,6}$  on vertices  $(u, v)$

Let the *path double star*  $\mathcal{PS}_{k,k}$  be a graph on  $2k+1$  vertices  $w, u, u_1, \dots, u_{k-1}, v, v_1, \dots, v_{k-1}$  such that  $E(\mathcal{PS}_{k,k}) = \{(u, u_i), 1 \leq i \leq k-1\} \cup \{(v, v_i), 1 \leq i \leq k-1\} \cup (v, w) \cup (u, w)\}$ , as shown in Figure 2, and let  $\mathcal{K}_{\mathcal{PS}_{k,k}}$  be the hypergraph of the game.

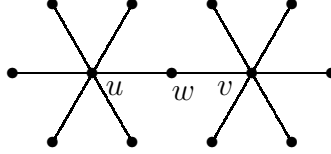


Figure 2:  $\mathcal{PS}_{6,6}$  on vertices  $(u, v, w)$

**Theorem 1.2.** *Let  $k \geq 2$ . In  $(1 : b)$  double star  $\mathcal{S}_{k,k}$  and path double star  $\mathcal{PS}_{k,k}$  games, we have*

$$(i) \quad f_{\mathcal{K}_{\mathcal{S}_{k,k}}}^{mon} = \Theta(n^{\frac{k}{k-1}}),$$

$$(ii) \quad f_{\mathcal{K}_{\mathcal{PS}_{k,k}}}^{mon} = \Theta(n^{\frac{k+1}{k}}).$$

The rest of the paper is organized as follows: In Section 2 we give some preliminaries and notation. In Section 3 we prove Theorem 1.1. In Section 4 we prove Theorem 1.2.

## 2 Preliminaries

Our graph-theoretic notation is standard and follows the one given in [17]. Wherever necessary, we suppose that  $n$  is sufficiently large. Throughout the paper  $G$  stands for

a graph with vertex set  $V(G)$  and the edge set  $E(G)$ . By  $v(G)$  and  $e(G)$  we denote the number of vertices, and number of edges respectively, in  $G$ . By  $A_i$ , respectively  $F_i$ , we denote the graphs with vertex set  $V(G)$ , whose edges are claimed by Avoider and Enforcer, respectively, in the first  $i$  turns. By  $d_{A_i}(v)$  and  $d_{F_i}(v)$  we denote degree of vertex  $v \in V(G)$  in Avoider's, respectively Enforcer's graph. Sometimes, we will refer to these parameters as  $A$ -degree and  $E$ -degree. The union of the two graphs  $A_i$  and  $F_i$  is the *global graph* and will be denoted by  $G_i$ . The set of *available* edges before the  $i$ th turn is  $E_i = E \setminus (E(A_{i-1}) \cup E(F_{i-1})) = E \setminus E(G_{i-1})$ .

The extremal number,  $ex(n, X)$ , is the maximum number of edges in a graph on  $n$  vertices which does not contain a copy of  $X$ .

Let us finish with two number theoretic statements that we will need in the analysis of strict games. The reason this is needed is the following: for fixed  $n$  the function  $r = r(n, b)$ , defined in introduction, behaves very strangely as  $b$  grows.

**Fact 2.1.** [5] Let  $r < 2$  be a rational number and  $c > 0$  be an integer. There are infinitely many natural numbers  $n$  such that

- (i)  $q \mid \binom{n}{2}$  for some  $q$  with  $cn^r < q < 4cn^r$ .
- (ii)  $q \mid \binom{n}{2} - 1$  for some  $q$  with  $cn^r < q < 4cn^r$ .

**Fact 2.2.** [5] For every  $\delta \in (0, 1)$  there exists an integer  $N_\delta$  such that if  $N \geq N_\delta$  and  $N^\delta < q < \frac{\delta N}{2 \log N}$  hold, then there exists an integer  $k$  such that  $q \leq k \leq 2q \log q / \delta$  and the remainder of the division of  $N$  by  $k$  is at least  $q$ .

## 3 The $k$ -star game – proof of Theorem 1.1

### 3.1 Enforcer's strategies – lower bounds

In this section we give lower bounds on threshold biases  $f_{\mathcal{K}_{S_k}}^{mon}, f_{\mathcal{K}_{S_k}}^+, f_{\mathcal{K}_{S_k}}^-$ . First, we describe the strategy of Enforcer in the monotone game and prove it is a winning strategy, thus establishing the lower bound on  $f_{\mathcal{K}_{S_k}}^{mon}$ .

#### Enforcer's strategy – the monotone game:

Enforcer's strategy is to enlarge his graph slowly whilst maintaining a clique in the global graph  $G$ . When a vertex with  $A$ -degree  $k - 1$  first appears outside the clique, Enforcer claims all unclaimed edges but one adjacent to  $v$ . So, after his move, there is either only one edge available in  $E$  that is adjacent to a vertex with  $A$ -degree  $k - 1$ , or there exists a partition  $V = I \cup C$  such that Enforcer's graph  $F[I]$  induced on  $I$  is empty and the global graph  $G[C]$  induced on  $C$  is a clique. Initially,  $I = V$  and  $C = \emptyset$ .

In his first turn, Enforcer is clearly able to establish the required property of his strategy. Indeed, if Avoider claimed  $k - 1$  edges adjacent to a vertex  $v$ , then Enforcer can claim all

edges but one adjacent to  $v$ . We will show that in this case, there are more than  $b$  unclaimed edges in the graph. If there is no vertex with  $A$ -degree at least  $k - 1$ , then Enforcer enumerates the vertices  $v_1^1, v_2^1, \dots, v_n^1$  in non-decreasing  $A$ -degree order and determines the smallest integer  $n_1$  such that  $\binom{n_1}{2} - e(A_1[v_1^1, v_2^1, \dots, v_{n_1}^1]) \geq b$  and claims all available edges joining each two of the first  $n_1$  vertices of this enumeration.

Let us suppose that for some  $i \geq 1$ , Enforcer was able to ensure the existence of  $I_i$  and  $C_i$ . Then in his  $(i+1)$ st turn, Enforcer checks if there is a vertex of  $A$ -degree at least  $k - 1$  in  $I_i$ , and if there is, he claims all but one edge incident to that vertex and wins provided there are still enough available edges to do so. If this is not the case, Enforcer enumerates the vertices  $v_1^{i+1}, v_2^{i+1}, \dots, v_{|I_i|}^{i+1}$  in non-decreasing  $A$ -degree order and determines the smallest integer  $n_{i+1}$  such that the number of available edges necessary for  $T_i[C_i \cup \{v_1^{i+1}, v_2^{i+1}, \dots, v_{n_{i+1}}^{i+1}\}]$  to be a clique (joining either two of the first  $n_{i+1}$  vertices of this enumeration or one of the first  $n_{i+1}$  vertices and a vertex from  $C_i$ ) is at least  $b$ . Then he claims all such edges and  $C_{i+1} = C_i \cup \{v_1^{i+1}, v_2^{i+1}, \dots, v_{n_{i+1}}^{i+1}\}$  and  $I_{i+1} = I_i \setminus \{v_1^{i+1}, v_2^{i+1}, \dots, v_{n_{i+1}}^{i+1}\}$  will possess the required property. Note that Enforcer does not claim more than  $b + n$  edges per move.

We have to show that if Enforcer plays according to this strategy, then a vertex of  $A$ -degree  $k - 1$  appears in  $I_i$  for some  $i$  and at this point the number of available edges is strictly larger than  $b$ . To analyse the strategy we divide the run of the game into stages. For  $j$ ,  $1 \leq j \leq k - 2$ , stage  $j$  ends in the  $i$ th turn, if  $\frac{1}{|I_i|} \sum_{v \in I_i} d_{A_i}(v) \geq j$  holds, i.e. the average  $A$ -degree in  $I_i$  is at least  $j$ . Note that if Avoider has to play at least one more move after the end of the stage  $j$ , then a vertex of  $A$ -degree  $j + 1$  appears. Thus at most  $k - 2$  stages can end without Avoider losing.

The following lemma estimates the size of  $I_i$  after stage  $j$ ,  $1 \leq j \leq k - 2$  is over. It takes into account the only thing we know about Avoider, that he claims at least one edge per move, thus increasing the  $A$ -degree of at least one vertex in  $I_i$  by at least one.

**Lemma 3.1.** *Let  $b \leq cn^{\frac{k}{k-1}}$  and  $b = \omega(n)$  hold, where  $c < \frac{1}{2}$ . Then for every  $j$  with  $1 \leq j \leq k - 2$ , stage  $j$  ends at some turn  $i = i(j)$  and the inequality  $|I_i| \geq \frac{1}{2} \frac{n^{j+1}}{(2b)^j}$  holds. Furthermore, for any  $b \leq cn^{\frac{k}{k-1}}$  with  $c < \frac{1}{2}$ , stage  $k - 2$  ends and at that moment  $|I| \geq \frac{1}{2} n^{\frac{1}{k-1}}$ .*

*Proof.* We proceed by induction. Let  $j = 1$  and suppose that after the  $i$ th turn we have  $|I_i| < \frac{1}{2} \frac{n^{j+1}}{(2b)^j}$  and stage one is still on, i.e.  $\frac{1}{|I_i|} \sum_{v \in I_i} d_{A_i}(v) < 1$  holds. Rearranging gives the inequality  $\sum_{v \in I_i} d_{A_i}(v) < \frac{1}{4} \frac{n^2}{b}$ . Because of Enforcer's strategy, in each turn  $l$ , Avoider must claim an edge having at least one of its end vertices, say  $u$ , in  $I_l$ , increasing  $u$ 's  $A$ -degree by one. As Enforcer moves vertices from  $I_l$  to  $C_{l+1}$  with minimum  $A$ -degree, it must be

that  $u \notin C_{l+1}$ . Indeed,  $u \in C_{l+1}$  would imply that all vertices in  $I_{l+1}$  have  $A$ -degree at least one and thus stage one should have ended. Therefore the sum  $\sum_{v \in I_l} d_{A_l}(v)$  increases by at

least one in each turn and thus  $i \leq \frac{1}{4} \frac{n^2}{b}$ . Each vertex in  $C_i$  is connected by Enforcer's edge to all, but at most  $k - 2$  vertices in  $C_i$ , so the number of Enforcer's edges in  $C_i$  is bigger than  $\frac{1}{2}(|C_i| - k)^2$ . On the other hand, Enforcer claims at most  $b + n$  edges per turn, hence  $\frac{1}{2}(|C_i| - k)^2 < i(b + n) < \frac{1}{4}n^2$ , thus  $|C_i| < \frac{\sqrt{2}}{2}n$  and  $|I_i| = n - |C_i| > \frac{2 - \sqrt{2}}{2}n$ , a contradiction.

Suppose that the Lemma holds for stage  $j - 1$ , and there exists  $i$ , such that  $|I_i| < \frac{1}{2} \frac{n^{j+1}}{(2b)^j}$  and stage  $j$  has not yet ended, i.e.  $\frac{1}{|I_i|} \sum_{v \in I_i} d_{A_i}(v) < j$  holds. For any turn  $l$  in stage  $j$  let

$w(l) = \sum_{v \in I_l} d_{A_l}(v) - (j - 1)|I_l|$ . As we are in stage  $j$ , this is a non-negative integer. Since the

degrees in Avoider's graph are increasing and Enforcer's strategy is to move the smallest  $A$ -degree vertices to  $C_{l+1}$  from  $I_l$ , we observe that  $w(l)$  is non-decreasing. Moreover, whenever the end vertex  $u$  of the edge claimed by Avoider is not moved to  $C_{l+1}$ , then  $w(l + 1)$  is strictly larger than  $w(l)$ . Note that if this is not the case, then the minimum  $A$ -degree in  $I_l$  is strictly less than the minimum  $A$ -degree in  $I_{l+1}$ , thus this can happen at most  $j - 1$  times. Note also that if  $w(l) \geq |I_l|$ , then the average  $A$ -degree in  $I_l$  is at least  $j$ , thus stage  $j$  would have ended. This yields  $i < \frac{1}{2} \frac{n^{j+1}}{(2b)^j} + j - 1$ .

On the other hand, as soon as  $|C_l| \geq 2n/3 + k$  holds, Enforcer, following his strategy, does not include more than  $\frac{3}{2} \frac{b}{n}$  new vertices into  $C_{l+1}$ . Thus, if stage  $j$  began at turn  $l_0$ , then we have  $|I_{l_0}| - |I_i| \leq \frac{3i}{2} \frac{b}{n} < \frac{3}{8} \frac{n^j}{(2b)^{j-1}}$ , but by induction  $|I_{l_0}| \geq \frac{1}{2} \frac{n^j}{(2b)^{j-1}}$ , which gives a contradiction.

The proof of the statement that holds for values of  $b$  of order  $O(n)$  is identical to the one above, the computations are a little different as in that case  $n^{j+1}/(2b)^j$  would yield an independent set  $I$  of size greater than  $n$ .  $\square$

By Lemma 3.1 after stage  $k - 2$ , there are at least  $\frac{1}{2}n^{\frac{1}{k-1}}$  vertices in  $I_i$ . As the appearance of a vertex of  $A$ -degree  $k - 1$  would have allowed Enforcer to win the game, after stage  $k - 2$  all vertices in  $I_i$  are of  $A$ -degree  $k - 2$ . With his next move Avoider creates a vertex with  $A$ -degree  $k - 1$  in his graph and he cannot claim more than  $|I_i|$  edges without creating a vertex of degree  $k$ . As  $|E_{i+1}| > |I_i|(|C_i| - k + 1)$  holds, the number of available edges at Enforcer's turn is more than  $|I_i||C_i| - k|I_i| > b$ , thus Enforcer may claim all edges but one adjacent to the vertex of  $A$ -degree  $k - 1$ .

The above strategy cannot be used in exactly the same form in the strict game for two reasons. First of all, as Enforcer must claim exactly  $b$  edges per move, he cannot make sure that he only claims edges that belong to a clique in the global graph. More importantly, even if Avoider creates a vertex  $v$  of  $A$ -degree  $k - 1$ , Enforcer cannot make sure he claims all edges but one incident to  $v$  to force Avoider to claim this edge and lose. In the remainder



of this subsection, we show how to overcome these difficulties.

**Enforcer's strategy – the strict game:**

Let us recall that  $1 \leq r \leq b+1$  denotes the integer with  $\binom{n}{2} \equiv r \pmod{b+1}$ . This is the number of edges Avoider will be able to choose from before his last move, and Enforcer will claim the remaining  $r-1$  edges in his last move. When there are at least  $r$  vertices  $v_1, \dots, v_r$  with  $A$ -degree  $k-1$  in Avoider's graph such that there are distinct available edges  $e_1, \dots, e_r$  with  $v_i$  incident to  $e_i$  (we will show later that this happens in the game), from then on Enforcer claims edges different from  $e_1, \dots, e_r$  and thus forces Avoider to claim one of the edges  $e_i$ ,  $1 \leq i \leq r$  in his last move (or even before) which will create an  $\mathcal{S}_k$  in Avoider's graph.

Until the appearance of the  $r$ th vertex of  $A$ -degree  $k-1$ , Enforcer follows his strategy in the monotone game with a slight modification. Enforcer's aim is that after his  $i$ th move,  $i \geq 1$ , there should exist a partition  $V = I_i \cup C_i$  such that Enforcer's graph  $F_i[I_i]$  induced on  $I_i$  is empty and the global graph  $G_i[C_i]$  induced on  $C_i$  is a clique. Initially,  $I_0 = V$  and  $C_0 = \emptyset$  as in the monotone game. The only difference compared to the situation in the monotone game is that Enforcer must claim exactly  $b$  edges. Therefore in his  $(i+1)$ st move,  $i \geq 0$ , he enumerates the vertices  $v_1^{i+1}, v_2^{i+1}, \dots, v_{|I_i|}^{i+1}$  of  $I_i$  in non-decreasing  $A$ -degree order and determines the largest integer  $n_{i+1}$  such that the number of available edges joining either two of the first  $n_{i+1}$  vertices of this enumeration or one of the first  $n_{i+1}$  vertices and a vertex from  $C_i$  is at most  $b$ . Then he claims all such edges and  $C_{i+1} = C_i \cup \{v_1^{i+1}, v_2^{i+1}, \dots, v_{n_{i+1}}^{i+1}\}$  and  $I_{i+1} = I_i \setminus \{v_1^{i+1}, v_2^{i+1}, \dots, v_{n_{i+1}}^{i+1}\}$  will possess the required property provided Enforcer can claim  $l_{i+1}$  more available edges that join vertices of  $I_{i+1}$  to  $C_{i+1}$  in order to claim exactly  $b$  edges. To do so, Enforcer picks the next  $2k$  vertices  $v_{n_{i+1}+1}^{i+1}, v_{n_{i+1}+2}^{i+1}, \dots, v_{n_{i+1}+2k}^{i+1}$  of the enumeration of  $I_i$  and for each  $1 \leq h \leq 2k$  he claims  $\lfloor \frac{l_{i+1}+h-1}{2k} \rfloor$  available edges (we call them 'extra edges') joining  $v_{n_{i+1}+h}^{i+1}$  to vertices of  $C_{i+1}$ . Also, note that when enumerating the vertices of  $I_i$  in his  $(i+1)$ st move, he always places vertices with extra edges as early as possible, that is among all the vertices with  $A$ -degree  $d$  first come those with extra edges and afterwards those without extra edges.

Now we prove that Enforcer is able to play according to the above strategy without ever having to forfeit the game.

The strategy in the strict game is similar to the strategy in monotone game. However, because Enforcer claims the extra edges in each his move, we need to prove he has enough edges to play the game till its end. Suppose, we are in the  $(i+1)$ st move of Enforcer,  $i \geq 0$  at the point before he adds  $l_{i+1}$  extra edges to the  $2k$  selected vertices from  $I_{i+1}$ . To see that there are enough available edges for all the vertices note that  $l_{i+1} < |C_{i+1}|$  by choice of  $n_{i+1}$  and clearly  $|C_j| \leq |C_{i+1}|$  holds for any  $j < i+1$ . Thus if this is the  $m$ th time that a vertex  $v$  is picked to receive extra edges in Enforcer's graph, then the total number of extra edges incident to  $v$  is at most  $m \lceil \frac{|C_{i+1}|}{2k} \rceil$ . By the way Enforcer enumerates vertices of



$I_i$ , the only way how a vertex  $v$  with extra edges can avoid being put to the clique  $G[C_{i+2}]$  in Enforcer's next move is that Avoider claims an edge incident to  $v$ . Thus, any vertex  $v$  can receive extra edges at most  $k$  times, as otherwise Avoider must have already claimed an  $\mathcal{S}_k$ . This means that when making his  $(i+1)$ st move Enforcer has at most  $k \lceil \frac{|C_{i+1}|}{2k} \rceil$  extra edges incident to any vertex  $v \in I_{i+1}$ . Thus, at any point of the game, there exist enough available edges between any vertex  $v \in I_{i+1}$  and  $C_{i+1}$  so that Enforcer is able to follow the above strategy. Moreover, at any turn  $j$ , during the game, every vertex  $v \in I_j$  has  $E$ -degree at most  $n/2$  and (unless Avoider has already lost) has  $A$ -degree at most  $k-1$ , thus there are at least  $n/4$  available edges in  $E_j$  incident to  $v$ .

Note that because  $G_j[C_j]$  is a clique for all  $j$ , in every turn Avoider must claim an edge with at least one endpoint in  $I_j$ . This means that Lemma 3.1 remains valid for the partitions  $I_j, C_j$  of the strict game. Let us consider the moment when stage  $k-2$  ends. Every vertex  $u \in I$  has  $A$ -degree at most  $k-1$  (otherwise Avoider has already lost) and, by Enforcer's strategy,  $E$ -degree at most  $|C|/2$ . Thus the number of available edges is at least  $\frac{1}{4}|I||C| \geq \frac{1}{8} \frac{n^k}{(2b)^{k-2}}$ , by Lemma 3.1. Therefore if the inequality  $r(b+1) < \frac{1}{8} \frac{n^k}{(2b)^{k-2}}$  holds, then till the end of the game there will be at least  $r+1$  more turns.

**Proposition 3.2.** *After Avoider  $l$ th move in stage  $k-1$  either Avoider's graph contains an  $\mathcal{S}_k$  or there are at least  $l$  vertices in  $I$  of  $A$ -degree  $k-1$ .*

*Proof.* Let  $v_1, v_2, \dots, v_t$  denote the vertices of  $I$  at the end of stage  $k-2$  with  $A$ -degree at most  $k-3$  and let us write  $m = \sum_{i=1}^t (k-2-d_A(v_i))$ . If Avoider has not yet created a  $\mathcal{S}_k$ , then all vertices have  $A$ -degree at most  $k-1$ , thus at the beginning of stage  $k-1$ , there are at least  $m$  vertices in  $I$  with  $A$ -degree  $k-1$ . As all edges claimed by Avoider during stage  $k-1$  have at least one endpoint in  $I$ , every move but at most  $m$  (joining either two  $v_i$ 's or a  $v_i$  to a vertex of  $C$ ) creates a new vertex of  $A$ -degree  $k-1$ .  $\square$

Proposition 3.2 shows that if the inequality  $r(b+1) < \frac{1}{8} \frac{n^k}{(2b)^{k-2}}$  holds, then after Avoider's  $r$ th move of the stage  $k-1$  there will be at least  $r$  vertices with  $A$ -degree  $k-1$  in  $I$ . Thus, Enforcer can use his strategy to force Avoider to lose. This proves  $e_{n,k}^+ \leq f_{\mathcal{K}_{S_k}}^+$  and  $e_{n,k}^- \leq f_{\mathcal{K}_{S_k}}^-$ . To see that the inequality  $\frac{1}{2} n^{\frac{k+1}{k}} \leq f_{\mathcal{K}_{S_k}}^-$  holds note that if  $b < \frac{1}{2} n^{\frac{k+1}{k}}$ , then by Lemma 3.1 even stage  $k-1$  ends and either Avoider has already lost or all vertices in  $I$  have  $A$ -degree  $k-1$  and Avoider loses in his next move.

Finally, applying Fact 2.1 (ii) with  $r = \frac{k}{k-1}$  and  $c = \frac{1}{8}$ , we obtain infinitely many integers  $n$  such that there exists an integer  $b$  with  $\frac{1}{8} n^{\frac{k}{k-1}} < b < \frac{1}{2} n^{\frac{k}{k-1}}$  and  $r = r(n, b) = 1$ . This shows that for these values of  $n$  we have  $e_{n,k}^+ \geq \frac{1}{8} n^{\frac{k}{k-1}}$ .

## 3.2 Avoider's strategies – upper bounds

Here, we establish upper bounds on threshold biases  $f_{\mathcal{K}_{S_k}}^{mon}, f_{\mathcal{K}_{S_k}}^+, f_{\mathcal{K}_{S_k}}^-$ . As we discussed in Section 3.1, in the monotone game Avoider is doomed at the moment he is forced to create

a vertex of  $A$ -degree  $k - 1$  and also in the strict game provided  $\binom{n}{2} \equiv 1 \pmod{b+1}$ . To prevent this situation Avoider will try to keep the maximum degree of his graph as low as possible with the following simple strategy.

**Avoider's rule:** Let  $E_i$  be the set of available edges in Avoider's  $i$ th turn, then for any edge  $(u, v) \in E_i$  let  $d_{\max}(u, v) = \max\{d_{A_i}(u), d_{A_i}(v)\}$ . Avoider picks an arbitrary edge from the set of available edges with minimum  $d_{\max}$ -value.

Note that if Avoider plays according to the above rule, then he claims exactly one edge in each of his steps, thus this is a valid strategy for both the monotone and the strict game. To obtain the upper bound of Theorem 1.1 (i) and (ii) we have to show that if Avoider plays according to this strategy, then he can win independent of Enforcer's strategy provided Enforcer's bias is at least  $2n^{\frac{k}{k-1}}$ . In order to be able to analyse how effective the above strategy is, just as in the previous subsection, we divide the run of the play into *stages* according to the maximum degree of Avoider's graph. This time, stage  $j$  starts with Avoider's move in which he creates the first vertex with  $A$ -degree  $j$  and ends with Enforcer's last move while Avoider's graph still has maximum  $A$ -degree  $j$ .

**Proposition 3.3.** *At the end of stage  $j$ , the vertices of  $A$ -degree at most  $j - 1$  must form a clique in the global graph.*

*Proof.* According to his strategy, Avoider would otherwise pick an edge without creating a vertex of  $A$ -degree  $j + 1$ .  $\square$

**Lemma 3.4.** *Assume that Enforcer's bias is  $b$ . Then for any positive integer  $j$ , at the end of stage  $j$  (if it ends), the number of vertices of  $A$ -degree  $j$  is at most  $(2^{j-1} + o(1))\frac{n^{j+1}}{b^j}$  and the number of still available edges is at most  $(2^{j-1} + o(1))\frac{n^{j+2}}{b^j}$ .*

*Proof.* We proceed by induction. As in every turn the total number of edges taken by Avoider and Enforcer is at least  $b + 1$ , the whole play, and thus stage 1, cannot last longer than  $\binom{n}{2}/(b + 1)$  turns. As in each turn Avoider creates two vertices of degree 1, the number of such vertices at the end of stage 1 is at most  $n(n - 1)/(b + 1) = (1 + o(1))\frac{n^2}{b}$ . By Proposition 3.3, none of the edges between two vertices of  $A$ -degree 0 is still available, therefore the number of available edges is at most  $(1 + o(1))\frac{n^2}{b} \cdot n = (1 + o(1))\frac{n^3}{b}$ .

Assume that the statement of the lemma holds for  $j$  and thus the number of available edges at the end of stage  $j$  is at most  $(2^{j-1} + o(1))\frac{n^{j+2}}{b^j}$ . Therefore stage  $j + 1$  cannot last longer than  $(2^{j-1} + o(1))\frac{n^{j+2}}{b^j(b+1)} = (2^{j-1} + o(1))\frac{n^{j+2}}{b^{j+1}}$ . Each turn Avoider takes exactly one edge, therefore in one turn at most two more vertices of  $A$ -degree  $j + 1$  can appear and thus the number of such vertices by the end of stage  $j + 1$  cannot exceed  $(2^j + o(1))\frac{n^{j+2}}{b^{j+1}}$ . Thus, by Proposition 3.3, the number of available edges at the end of stage  $j + 1$  is at most  $(2^j + o(1))\frac{n^{j+2}}{b^{j+1}} \cdot n = (2^j + o(1))\frac{n^{j+3}}{b^{j+1}}$ .  $\square$

Lemma 3.4 easily implies the upper bounds of Theorem 1.1 on  $f_{\mathcal{K}_{S_k}}^{mon}$  and  $f_{\mathcal{K}_{S_k}}^+$ . Indeed, if Enforcer's bias is at least  $2n^{\frac{k}{k-1}}$ , then if Avoider uses the above strategy, then either stage  $k-2$  does not end during the run of the game and thus not even a vertex of  $A$ -degree  $k-1$  is created, or at the end of stage  $k-2$  the number of still available edges is strictly less than  $b+1$ , the minimum number of edges taken in any turn. Therefore no matter which edge Avoider picks in the first turn of stage  $k-1$ , Enforcer will claim all remaining edges in his first turn in stage  $k-1$ .

To see the upper bound on  $f_{\mathcal{K}_{S_k}}^-$  let us suppose that Enforcer's bias  $b$  is  $2n^{\frac{k+1}{k}} < b < 8n^{\frac{k+1}{k}}$  and  $\binom{n}{2} \equiv 0 \pmod{b+1}$  holds. According to Fact 2.1 (i), there exists infinitely many integers  $n$  with such  $b$ . Then applying Lemma 3.4 shows that regardless of Enforcer's strategy at the end of stage  $k-2$ , the number of vertices of  $A$ -degree  $k-2$  is at most  $\frac{1}{2}n^{\frac{2}{k}}$  and the number of available edges is at most  $\frac{1}{2}n^{\frac{k+2}{k}}$ . Therefore there remain at most  $\frac{1}{4}n^{\frac{1}{k}}$  turns in the game. In each step Avoider creates at most 2 vertices of  $A$ -degree  $k-1$ , thus even before Avoider's final turn, there is at most  $\frac{1}{2}n^{\frac{1}{k}}$  of them. To prevent himself from creating a vertex of  $A$ -degree  $k$ , Avoider must not take at most  $\frac{1}{2}n^{\frac{k+1}{k}}$  still available edges incident to these vertices. But as  $\binom{n}{2} \equiv 0 \pmod{b+1}$  holds, Avoider in his last turn has the possibility to choose among  $b \geq 2n^{\frac{k+1}{k}}$  available edges, which is more than the number of edges he should avoid.

Finally, applying Fact 2.2 with  $N = \binom{n}{2}$ ,  $\delta = \frac{k+1}{2k}$  and  $q = 2n^{\frac{k+1}{k}}$  we obtain that for any sufficiently large  $n$  there exists an integer  $b$  with  $2n^{\frac{k+1}{k}} \leq b \leq 8n^{\frac{k+1}{k}} \log n$  such that the remainder  $r = r(n, b)$  is at least  $2n^{\frac{k+1}{k}}$ . A computation identical to the one above yields the general statement about the upper bound on  $f_{\mathcal{K}_{S_k}}^-$ .

## 4 Monotone double star games

In this section we show how to modify Enforcer's basic strategy for the monotone degree game to obtain a strategy in the double star and the path double star games.

### 4.1 Proof of Theorem 1.2 (i)

First, we will give Enforcer's strategy and then prove he can follow it and also that it is indeed a winning strategy.

#### **Enforcer's strategy in the monotone $\mathcal{S}_{k,k}$ game:**

Phase I: Until a vertex  $v$  of  $A$ -degree at least  $k-1$  or an  $\mathcal{S}_{k,k}$  is created, Enforcer follows the strategy he used in the  $\mathcal{S}_k$ -game. If an  $\mathcal{S}_{k,k}$  is created then Enforcer won the game already. If there is vertex  $v$  that has  $A$ -degree at least  $k-1$ , but there is no  $\mathcal{S}_{k,k}$  in  $A$ , then he performs a switching move.

Switching move: Vertex  $v$  has  $A$ -degree at least  $k-1$  and it is the vertex with largest  $A$ -degree. Enforcer chooses some  $k-1$  neighbours of  $v$  arbitrarily, we call them  $n_1, n_2, \dots, n_{k-1}$  and for each  $n_j$ ,  $1 \leq j \leq k-1$ , he chooses all its neighbours. Let  $N$  denote the set of those  $(k-1)^2$  vertices. Enforcer then claims all edges available that are adjacent to every vertex from  $N$ . Also, Enforcer moves  $v$  from  $I$  to  $C$  and claims the available edges between  $v$  and every vertex from  $C$ . This is the end of Phase I and he proceeds to the Phase II.

Phase II: Enforcer uses the strategy of the  $\mathcal{S}_k$ -game with the slight modification that he maintains a tripartition  $V = N \cup I \cup C$  such that  $G[C]$  is a clique,  $F[I]$  and  $F[I, C]$  are empty. When another vertex  $u \in I$  of  $A$ -degree  $k-1$  is created while the number of available edges is more than  $b$  (we will show this later), then Enforcer wins the game. Since  $v$  is not in  $I$  after the switching move, Enforcer does not claim any edge incident to it. Also,  $u \in I$ , and according to above said,  $F[I]$  is empty. Hence, by Enforcer's strategy the edge  $(u, v)$  is either already claimed by Avoider or is still available. In the former case Enforcer claims all edges but one joined to  $u$ , in the latter he claims all edges but  $(u, v)$ . In both cases, Avoider is forced to complete a  $\mathcal{S}_{k,k}$ , thus losing the game.

This is the strategy, and now we will show Enforcer can follow it without ever having to forfeit the game.

According to the Lemma 3.1, Phase I ends. Let  $v_1, v_2, \dots, v_{k-1}$  denote the  $A$ -neighbours of  $v$ . The switching move guarantees that from then on, no vertex in  $I$  is connected to any of the  $v_j$ ,  $j \leq k-1$ .

By the above argument, all we have to prove is that during Phase II a new vertex of  $A$ -degree  $k-1$  is created with more than  $b$  available edges remaining provided  $b \leq cn^{\frac{k}{k-1}}$ , where  $c < \frac{1}{4k^3}$ . We distinguish cases according to the stage of Phase I during which the first vertex  $v$  of  $A$ -degree  $k-1$  appeared.

Let us first consider the case when the  $A$ -degree of  $v$  reached  $k-1$  *after* the end of stage  $k-2$ . By the discussion of Section 3, we know that in this case all vertices in  $I$  have  $A$ -degree  $k-2$ . Thus any edge claimed by Avoider will create at least one vertex  $u$  of  $A$ -degree  $k-1$ . Therefore all we have to verify is that the number of available edges, when Enforcer makes his second move after the end of stage  $k-2$  (i.e. the first non-switching move), is larger than  $b$ . By Lemma 3.1, at the end of stage  $k-2$  we have  $|I| \geq \frac{1}{2}n^{\frac{1}{k-1}}$  and thus the number of available edges is at least  $(\frac{1}{2} - o(1))n^{\frac{k}{k-1}}$ . As Enforcer never claims more than  $b + n$  edges, he will either have many available edges at his second turn after the end of stage  $k-2$  or Avoider has to claim  $(\frac{1}{2} - o(1))n^{\frac{k}{k-1}} - 2b - n = \Omega(n^{\frac{k}{k-1}})$  edges. But in this case he created an  $\mathcal{S}_{k,k}$  as  $ex(n, \mathcal{S}_{k,k}) \leq (k-1 + o(1))n$ .

Suppose next that  $v$  reached  $A$ -degree  $k-1$  during stage  $j$  for some  $1 \leq j \leq k-2$ . We want to show that stage  $j$  ends after at most  $k^3$  extra moves compared to what happens in the  $\mathcal{S}_k$ -game. Let us consider the weight function we used in the proof of Lemma 3.1,

$w(i) = \sum_{x \in I_i} d_{A_i}(x) - (j-1)|I_i| = \sum_{x \in I_i} (d_{A_i}(x) - (j-1))$ . An important observation in the proof of Lemma 3.1 was that  $w(i)$  is non-decreasing during stage  $j$ . This is not the case any more as the value can decrease in the switching move. The vertices with  $A$ -degree larger than  $j-1$  that are removed from  $I$  are  $v$  and the vertices of  $N$ , as otherwise all remaining vertices have  $A$ -degree  $j$  and stage  $j$  ends immediately. For each vertex  $u \in \{v\} \cup N$ , the summand in  $w(i)$  that corresponds to  $u$  is  $d_{A_i}(u) - (j-1) \leq k$  and hence the weight cannot decrease by more than  $k^3$ . Thus, stage  $j$  ends in at most  $k^3$  more turns than in the  $\mathcal{S}_k$ -game.

Let us assume that  $j = k-2$ . As Enforcer never claims more than  $b+n$  edges at a turn, he claims at most  $k^3(b+n) \leq (\frac{1}{4} + o(1))n^{\frac{k}{k-1}}$  edges more than in stage  $k-2$  of the  $\mathcal{S}_k$ -game. Therefore the number of available edges at the end of stage  $k-2$  is at least  $(\frac{1}{4} - o(1))n^{\frac{k}{k-1}}$  and thus either Avoider claims  $\Omega(n^{\frac{k}{k-1}})$  edges in his next move completing an  $\mathcal{S}_{k,k}$  or before the following Enforcer's move the number of available edges will still be more than  $b$ .

Finally, let us assume that  $1 \leq j \leq k-3$  holds. Then if  $|I| \leq n/2$  at each extra step Enforcer removes at most  $\frac{b+n}{\frac{n}{2}-j}$  vertices from  $I$  and thus the order of the size of the set of removed vertices in the extra  $k^3$  turns is negligible compared to  $\frac{1}{2} \frac{n^{j+1}}{(2b)^j} \geq n^{\frac{k-1-j}{k-1}}$ , the order of the size of  $I$  at the end of stage  $j$  in the  $\mathcal{S}_k$ -game. Therefore the number of vertices in  $I$  at the end of stage  $j$  in the  $\mathcal{S}_{k,k}$ -game is at least  $c'n^{\frac{k-1-j}{k-1}}$  for any  $c' < 1/2$ . The calculation for all later stages is identical to that in Lemma 3.1, therefore we obtain that for any  $c' < 1/2$ , stage  $k-2$  ends and at that moment the number of vertices in  $I$  is at least  $c'n^{\frac{1}{k-1}}$ , thus Enforcer wins within the next two moves.

## 4.2 Proof of Theorem 1.2 (ii)

### Enforcer's strategy in the monotone $\mathcal{PS}_{k,k}$ game:

We show that the Enforcer's strategy to win the  $\mathcal{S}_{k+1}$  game with bias  $b$  can be modified to obtain a winning strategy for the  $\mathcal{PS}_{k,k}$  game with bias of the same order of magnitude as  $b$ . First, we will describe the strategy and afterwards prove that it is indeed the winning strategy. Let  $b < cn^{\frac{k+1}{k}}$ , where  $c = c(k) = \frac{1}{16(k+3)^5}$ .

Before we describe the strategy of Enforcer, let us introduce some terminology. By  $\Gamma_A(v)$ , we denote the set of neighbours of vertex  $v$  in Avoider's graph  $A$ . We use the notation from the proof of monotone  $k$ -star game, namely sets  $I$  and  $C$ . We also define sets *centres*,  $X$ , and *neighbours*,  $N$ . The sets  $X$  and  $N$  determine a modification of the  $A$ -degree which we call  $A^*$ -degree that will influence which vertices Enforcer adds to  $C$ .

The  $A^*$ -degree of vertex  $v$  is:

$$d^*(v) := |\Gamma_A(v) \setminus (N \cup X)|.$$

Also, let  $d^*(I) = \sum_{v \in I} d^*(v)$ .

This new strategy of Enforcer requires us to define the *updating move*.

The *updating move*: Suppose  $|X| = h - 1$  for some  $h$ . If there exists at least one vertex  $v \in I$  s.t.  $d^*(v) \geq k$ , then choose the vertex  $v$  whose  $d^*(v)$  is maximal and label it by  $c_h$ . Then choose  $k$  vertices of the lowest  $d^*$  from  $\Gamma_A(v) \setminus (N \cup X)$  and label them  $n_h^1, \dots, n_h^k$ . Now *update*  $X$  and  $N$  by setting  $X = X \cup \{c_h\}$ ,  $N = N \cup \{n_h^1, \dots, n_h^k\}$  and  $I = I \setminus \{v\}$ .

Note if two vertices,  $u, v$  in Avoider's graph simultaneously reach  $A^*$ -degree  $k$  by Avoider's edge  $uv$ , then Enforcer arbitrarily adds one of them to  $X$ , say  $u$ , meaning  $uv$  no longer contributes to  $d^*(v)$  and so the  $A^*$ -degree of  $v$  is back to  $k - 1$ . Thus Enforcer adds at most one vertex to  $X$  after any Avoider's edge. Before any *updating* move is played the  $A^*$ -degree is equal to the  $A$ -degree.

**Enforcer's strategy.** During the course of the game, Enforcer maintains the sets  $X$ ,  $N$ ,  $I$  and  $C$ . The vertex set of the global graph is a tripartition  $V = I \cup C \cup X$  and  $N \subseteq I$ . Initially,  $X = \emptyset$ ,  $N = \emptyset$ ,  $C = \emptyset$  and  $I = V$ . Moreover, Enforcer will take care that throughout the game  $F[I]$  and  $F[I, C]$  are empty and  $G[C]$ ,  $G[X]$ ,  $G[X, V \setminus X]$  are complete graphs.

Enforcer follows the strategy for  $\mathcal{S}_{k+1}$  with a small modifications. Before each his move he does the following:

- (i) If  $\mathcal{PS}_{k,k}$  is a subgraph of  $A$ , Avoider has lost the game and Enforcer can claim all the remaining edges to finish the game.
- (ii) If  $\mathcal{PS}_{k,k} - e$  is a subgraph of  $A$ , with  $e$  unclaimed, then Enforcer takes all the remaining edges except for  $e$ .
- (iii) If not, then for any vertex  $x \in I$  such that  $d^*(x) \geq k$ , Enforcer performs the updating move and claims all unclaimed edges incident with  $x$ .
- (iv) With any remaining edges Enforcer plays according to the strategy in  $\mathcal{S}_{k+1}$ , but taking into consideration  $A^*$ -degree. Namely, he repeatedly chooses a vertex  $v \in I$  of minimum  $A^*$ -degree, connecting it to all the vertices in  $C$  and afterwards removing  $v$  from  $I$  and adding it to  $C$ .

We show that this is a winning strategy for Enforcer in two steps. Firstly, in Lemma 4.1, we show that  $|X| \geq k + 2$  will guarantee the appearance of a  $\mathcal{PS}_{k,k} - e$  with  $e$  unclaimed. Secondly we show that following the given strategy Enforcer can always guarantee that



$|X| \geq k + 2$  with at least  $b + 1$  remaining edges. After that, Enforcer claims all edges, but  $e$  forcing Avoider to lose.

**Lemma 4.1.** *If  $|X_i| \geq k + 2$ , then at turn  $i$  or at some earlier turn either  $\mathcal{PS}_{k,k}$  or  $\mathcal{PS}_{k,k} - e$  is a subgraph of  $A$ , where  $e$  is an unclaimed edge.*

*Proof.* Since  $|X_i| \geq k + 2$  we must have some vertex  $c_j \in X_i$  such that  $c_j \notin \{c_1, n_1^1, \dots, n_1^k\}$ . Consider the moment the *updating move* was played and  $c_j$  was added to the set  $X$ . Suppose it was in turn  $p$ ,  $p \leq i$ . Then  $X_p \subseteq X_i$ . Case 1: For some  $l$ ,  $n_1^l \notin X_p$ , then set  $e = n_1^l c_j$ . Observe that by Enforcer's strategy, every Enforcer's edge is either incident to a vertex in  $X$  or lies wholly within  $C$ . Hence before the *updating move*, either  $e \in A$  or  $e$  is unclaimed and so there is a  $\mathcal{PS}_{k,k} - e \in A$  on vertices  $(c_1, n_1^l, c_j)$ . Case 2: For all  $l$ ,  $1 \leq l \leq k$ ,  $n_1^l \in X_p$ . Fix some  $h, h' < j$ ,  $c_h, c_{h'} \in \{n_1^1, \dots, n_1^k\}$ . In particular  $c_1 c_h \in A$  and  $c_1 c_{h'} \in A$ . Recall that by construction the  $k$  neighbours of  $c_{h'}$  and  $c_h$ , i.e. the sets  $\{n_h^1, \dots, n_h^k\}$  and  $\{n_{h'}^1, \dots, n_{h'}^k\}$ , are disjoint. Therefore, there exists a  $\mathcal{PS}_{k,k} \in A[G]$  on vertices  $(c_h, c_1, c_{h'})$  as required.  $\square$

Now we need to show that following the given strategy, Enforcer ensures  $|X| \geq k + 2$  while there are at least  $b + 1$  unclaimed edges remaining.  $A_i$  and  $F_i$  denote the  $i$ th move of Avoider, respectively Enforcer. Before that, let us analyse the course of the game when the updating move is played. When a vertex  $c_j$  is added to  $X$ , then for any vertex  $v \in I$ ,  $d^*(v)$  decreases by  $|\Gamma_A(v) \cap \{c_j, n_j^1, \dots, n_j^k\}|$ . By definition, at this point each of  $n_j^1, \dots, n_j^k$  have  $A^*$ -degree at most  $k$ . This follows from the fact that when  $c_j$  is chosen to be added to  $X$ , it has  $A^*$ -degree at least  $k$ . If  $A^*$ -degree is exactly  $k$ , then all of its neighbours can have  $A^*$ -degree at most  $k$  (otherwise,  $c_j$  would not be chosen to be added to  $X$ ). If  $A^*$ -degree of  $c_j$  is greater than  $k$ , then by the way of choosing  $n_j^1, n_j^2, \dots, n_j^k$ , having  $d^*(n_j^p) > k$ ,  $1 \leq p \leq k$  would mean that there exists a vertex  $u$  s.t. edge  $c_j, u$  is in  $A$  and thus  $A$  contains a  $\mathcal{PS}_{k,k} - e$  already. So, each of  $n_j^1, n_j^2, \dots, n_j^k$  has less than  $k + 1 + |N| + |C|$   $A^*$ -neighbours. Observe that  $|X| < k + 2$ , implies  $|N| < k(k + 2)$  and  $|E(N)| < \binom{k+2}{2}^k$ . Hence, the sum of  $d^*(v)$ ,  $v \in I$ , decreases by at most  $(k + 2)(k + 3)(k + 1) < (k + 3)^3$ . Also note that the only edges Avoider can place that do not increase the  $A^*$ -degree of at least one vertex in  $I$  have both endpoints in  $N$ .

By  $d_{A_i}^*(v)$ , respectively  $d_{F_i}^*(v)$ , we denote  $A^*$ -degree of a vertex  $v$  right after Avoider, respectively Enforcer, has played his  $i$ th move.

For any edge  $e$  added by Avoider in turn  $i$  either

- (i)  $d_{A_i}^*(I_{i-1}) \geq d_{F_{i-1}}^*(I_{i-1})$  and  $X, N$  unchanged,
- or (ii)  $d_{A_i}^*(I_{i-1}) \geq d_{F_{i-1}}^*(I_{i-1}) - (k + 3)^3$  and  $|X|$  increases by one.

By Enforcer's strategy at the beginning of Avoider's turn there are no unclaimed edges incident to vertices  $X$ . So, for the edge  $e$  that Avoider plays in turn  $i$ , it holds:

for  $e \notin E(N_i)$ ,

- (i)  $d_{A_i}^*(I_{i-1}) > d_{F_{i-1}}^*(I_{i-1})$  and  $X, N$  unchanged,
- or (ii)  $d_{A_i}^*(I_{i-1}) \geq d_{F_{i-1}}^*(I_{i-1}) - (k + 3)^3$ ,  $|X_i| = |X_{i-1}| + 1$  and  $|N_i| = |N_{i-1}| + k$ .



for  $e \in E(N_i)$ ,

(iii)  $d_{A_i}^*(I_{i-1}) = d_{F_{i-1}}^*(I_{i-1})$  and  $X, N$  unchanged,  $E(N_i) = E(N_{i-1}) - 1$ .

In the proof of Lemma 3.1 we argued that the function  $w$ , depending on  $A$ -degree over  $I$ , never decreases and must strictly increase on all but at most  $k$  of Avoider turns each stage. In stage  $j$ , let  $w^*(i) = \sum_{v \in I_i} \min\{k, d_{A_i}^*(v)\} - (j-1)|I_i|$ . We need to take min of the two values to prevent ending stage  $j$  with few vertices that have very large  $A^*$ -degree and the remaining vertices with  $A^*$ -degree less than  $k-2$ . We show that this function in  $\mathcal{PS}_{k,k}$  behaves similarly to  $w$  in  $\mathcal{S}_{k+1}$ .

Thus,  $w^*$  increases in all but at most  $f(k) = \binom{(k+2)k}{2}$  turns, when an edge from  $E(N)$  is claimed. Each time the set  $X$  is enlarged by some  $q \geq 1$  new vertices,  $w^*$  decreases by less than  $q \cdot g(k)$ , where  $g(k) = (k+3)^3$ . This implies that we should allow at most  $m(k) = k+2 + f(k) + (k+3) \cdot g(k)$  more turns each stage.

We distinguish cases according to the stage of the strategy for  $\mathcal{S}_{k+1}$ , when the first updating move occurred.

Let us first consider the case when the first updating move occurred after the end of stage  $k-1$ . By the discussion in Section 3.1, we know that in this case all the vertices in  $I$  have  $A$ -degree and  $A^*$ -degree  $k-1$ . Hence, any edge added by Avoider, creates a vertex of  $A$ -degree  $k$ . At the end of stage  $k-1$ ,  $|I| \geq \frac{1}{2}n^{\frac{1}{k}}$  and the number of available edges is at least  $(\frac{1}{2} - o(1))n^{\frac{k+1}{k}}$ . To achieve that  $|X| \geq k+2$ , we need at most  $m(k)$  more turns in the game. As Enforcer never claims more than  $b+n$  edges per turn, there are either lots of available edges remaining when  $|X| \geq k+2$  or Avoider has claimed at least  $(\frac{1}{2} - o(1))n^{\frac{k+1}{k}} - (m(k)+1) \cdot b - m(k) \cdot n = \Omega(n^{\frac{k+1}{k}})$  edges in that stage. However, this means that Avoider has already claimed a  $\mathcal{PS}_{k,k}$ , since  $ex(n, \mathcal{PS}_{k,k}) \leq (k+o(1))n$ .

Let us now consider the situation when the first updating move occurred during the stage  $k-1$ . In this situation, Enforcer claims no more than  $m(k) \cdot (b+n) \leq (\frac{1}{4} + o(1))n^{\frac{k+1}{k}}$  additional edges in stage  $k-1$ . At the time stage  $k-1$  ended, at most  $k^2$  vertices in  $I$  can have  $A^*$ -degree  $k-2$ . Otherwise, there would be more than  $k^2$  vertices of  $A^*$ -degree  $k$ , which would imply the existence of either  $\mathcal{PS}_{k,k}$  or  $\mathcal{PS}_{k,k} - e$ , with  $e$  unclaimed, in  $A$ . This implies that there are at least  $(\frac{1}{4} - o(1))n^{\frac{k+1}{k}}$  available edges after the end of stage  $k-1$ . Thus, either Avoider afterwards claims  $\Omega(n^{\frac{k+1}{k}})$  edges and loose, since his graph contains a  $\mathcal{PS}_{k,k}$  or there are enough available edges to continue playing at least  $m(k) + k^2$  turns in this stage.

If the first updating move occurred in stage  $j$ ,  $1 \leq j \leq k-2$ , the bias  $b$  ensures  $m(k)$  extra turns for Avoider each stage in the  $\mathcal{PS}_{k,k}$  game, than in the  $\mathcal{S}_{k+1}$  game. If  $I \geq \frac{n}{2}$ , the number of vertices removed in each extra turn is at most  $\frac{b+n}{\frac{n}{2} - (k+2)^2}$ . Hence the number of vertices removed in at most  $(k-1) \cdot m(k)$  is negligible compared to the size of  $I$  after

each stage  $j$  in  $\mathcal{S}_{k+1}$  game, which is  $\frac{1}{2} \frac{n^{j+1}}{(2b)^j} \geq (k+2)^{5j} n^{\frac{k-j}{k}}$ . For any  $c' < \frac{1}{2}$ , the number of vertices in  $I$  at the end of each stage  $j$  is at least  $c'(k+2)^{5j} n^{\frac{k-j}{k}}$ . Thus, stage  $k-1$  ends while the number of available edges in the graph is at least  $c'((k+2)^{5k-5} - o(1))n^{\frac{k+1}{k}}$ . If  $|X| \geq k+2$  at that point, Enforcer forces Avoider to lose as there are at least  $b+1$  unclaimed edges remaining. Otherwise, after  $m(k) + k^2$  more Avoider turns at least  $k+2$  more vertices that have  $A^*$ -degree  $k$  will be created and added to  $X$ . At that moment,  $A$  will contain either  $\mathcal{PS}_{k,k}$  or  $\mathcal{PS}_{k,k} - e$ . But after  $m(k) + k^2$  turns there are still at least  $b+1$  free edges and so Enforcer claims all the remaining edges except  $e$  and Avoider loses the game.

### Avoider's strategy in the monotone $\mathcal{PS}_{k,k}$ game:

In our strategy for Avoider to win the  $\mathcal{S}_{k+1}$  game with bias  $b$ , we showed that if at any stage  $i$ ,  $1 \leq i \leq k-1$ , there are less than  $b+1$  unclaimed edges remaining, he can avoid creating an  $\mathcal{S}_{k+1}$ . Moreover, following this strategy Avoider has created no vertices of degree  $k$  whenever there are at least  $b+1$  edges remaining, including after his penultimate turn. Hence in his last move he cannot be forced to create a  $\mathcal{PS}_{k,k}$ .

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